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Turbomachinery Application of Lagrangian Dynamics to the Motion of Continuous Discrete Rotors

Abstract

The stability/instability condition of a turbine rotor with axisymmetric supports is determined in the presence of gyroscopic loads and rub-induced destabilizing forces. A modal representation of the turbine engine is used, with one mode in each of the vertical and horizontal planes. The use of non-spinning rotor modes permits an explicit treatment of gyroscopic effects. The two linearized modal equations of motion of a rotor with axisymmetric supports are reduced to a single equation in a complex variable. The resulting eigenvalues yield explicit expressions at the stability boundary, for the whirl frequency as well as the required damping in the presence of the available rub-induced de-stabilization. Conversely, the allowable destabilization in the presence of the available damping is also given.

Introduction

Simulation of blade to case rubbing is critical to the complete understanding of the dynamic behavior of rotordynamic systems such as aircraft engines. Aircraft engines experiencing unbalance loadings, blade-outs, or mechanical misalignments may incur blade to case rubbing which may adversely affect the dynamic behavior of the engine and supporting structures. In some situations, the rubbing may become excessive and may even lead to structural instability and in severe cases catastrophic failure of the structure.

The purpose of the present paper is to determine the conditions for stability/instability of a turbine engine with rub-induced forces in the presence of gyroscopic forces. To develop a direct determination of the stability/instability conditions, the equations of motion with rub forces obtained in Reference [1] are used as the starting point for the present investigation. In this study, the vertical and horizontal planes are each represented with one non-spinning vibration mode. Stability/instability is then investigated by examining the corresponding eigenvalues of the characteristic equation.

The thrust of this work is with a rotor with axisymmetric support stiffness. This model allows for a complex-variable representation of the rotor system, and a more direct determination of conditions at the stability-instability boundary. The notion of neutral stability at the boundary permits obtaining two real equations from the complex eigenvalue problem. Equations are derived for the possible whirl frequencies and the required damping (or the allowable de-stabilization), at the stability/instability boundary.

Blade Rub Forces and Moments

Limited data and experience with rub tests on turbofan engines and test rigs indicate that blade tip rub forces, resulting from blade tip and case interactions, are in a direction that is perpendicular to the blade tip chordline. This phenomenon is similar to the behavior of an airfoil

traveling through a fluid where the airfoil motion generates a lift and drag forces. From tests, the tangential rub blade force was found to have an axial component that would move the blade forward (or aft), and a circumferential component that opposes the rotor rotation. The hydrodynamic analogy expresses the blade rub force as proportional to the blade tip's mean velocity and the instantaneous angle of attack or incidence. It differs from the typically used friction model of rub in that the friction model only generates a force in the direction opposing the rotor spin (i.e., drag) while the plowing model generates this drag force, as well as a "lifting" force that is perpendicular to the direction of rotor spin. A more detailed description of this rub model is provided in Reference [1].

An analytical expression, analogous to the dynamic pressure force from a change in momentum is used to develop the plowing model. This model is consistent with expressions of impact forces and hydrodynamic pressure. This was necessitated by the observed directionality of the blade tip "tangential" rub forces, i.e., perpendicular to the blade chordline.

The hydrodynamic force is defined as:

$$\bar{f} = D\dot{\psi}^2 R^2 \gamma_0 A = DV^2 \gamma_0 A \tag{1}$$

where $\bar{f}$ is the total tangential force on the blade from rubbing, V is the blade tip velocity, γ_0 is the blade chordline angle of incidence in the rotating frame, and A is the frontal area of the rubbing material (i.e., chord length x rub depth). D is a coefficient that must be determined from test or some other analysis procedure. It is most likely a function of several factors such as the material's Young and Shear moduli, density, hardness, machinability, feed depth, dynamic shear strength, smoothness and probably others.

The loads in the fixed coordinates system are:

$$\bar{f}_x = \bar{f} \sin\gamma_0 \sin\beta - \overline{K}_r x \tag{2a}$$

$$\bar{f}_y = -\bar{f} \sin\gamma_0 \cos\beta - \overline{K}_r y \tag{2b}$$

$$\bar{f}_a = -\bar{f} \cos\gamma_0 \cos\beta R \tag{3a}$$

$$\bar{f}_\theta = \bar{f} \cos\gamma_0 \sin\beta R \tag{3b}$$

Where $\overline{K}_r$ is the radial stiffness. Noting that:

$$\sin\beta = \frac{y}{r} \text{ and } \cos\beta = \frac{x}{r}$$

And substituting into (2) and (3), the rub loads become,

$$\bar{f}_x = D\dot{\psi}^2 R^2 \gamma_0 y \sin\gamma_0 - \bar{K}_r x \tag{4a}$$

$$\bar{f}_y = -D\dot{\psi}^2 R^2 \gamma_0 x \sin\gamma_0 - \bar{K}_r y \tag{4b}$$

$$\bar{f}_\alpha = -D\dot{\psi}^2 R^3 \gamma_0 x \cos\gamma_0 \tag{5a}$$

$$\bar{f}_\theta = D\dot{\psi}^2 R^3 \gamma_0 y \cos\gamma_0 \tag{5b}$$

Modal Transformation of the Rub Loads

As a first step to simplifying the equations of motion, the translational and rotational rub forces in (4) and (5) are transformed to modal coordinates. The following considerations are used to obtain the modal rub forces:

1. There is one modal function in each of the horizontal (x, α) and vertical (y, θ) planes.

2. The rotational (α, θ) deflections are the slopes of the bending displacements.

3. Because of the (x, y, α, θ) sign conventions, it is seen that:

$$\theta = -\frac{\partial y}{\partial z} \quad \text{and} \quad \alpha = +\frac{\partial x}{\partial z}$$

4. The modal functions in the fixed axes are non-rotating.

5. The modal functions are mass normalized.

Now let the modal displacements be as follows:

$$x = \bar{x}h(t) \tag{6a}$$

$$y = \bar{y}v(t) \tag{6b}$$

$$\alpha = \bar{\alpha}h(t) \tag{7a}$$

$$\theta = -\bar{\theta}v(t) \tag{7b}$$

Where $\bar{x}$, $\bar{y}$, $\bar{\alpha}$, $\bar{\theta}$ are mode shapes and h(t) and v(t) are horizontal and vertical plane displacement generalized coordinates, respectively.

Following the Principle of Virtual Work, the modal forces in the horizontal and vertical planes are:

$$\bar{f}_h = \int \left(\bar{x} \bar{f}_x + \bar{\alpha} \bar{f}_\alpha \right) ds \qquad (8a)$$

$$\bar{f}_v = \int \left(\bar{y} \bar{f}_y - \bar{\theta} \bar{f}_\theta \right) ds \qquad (8b)$$

Where the rub forces are integrated over the tangential extent of the rub, "s", then summed over all rubbing blades.

Making the physical-to-modal displacement substitutions, the modal rub forces become:

$$\bar{f}_h = \int \left\{ D\dot{\psi}^2 R^2 \gamma_0 \left(\bar{x}\bar{y} \sin \gamma_0 v(t) - R\bar{\alpha}\bar{x} \cos \gamma_0 h(t) \right) - \bar{K}_r \bar{x}^2 h(t) \right\} ds \qquad (9a)$$

$$\bar{f}_v = \int \left\{ - D\dot{\psi}^2 R^2 \gamma_0 \left(\bar{x}\bar{y} \sin \gamma_0 h(t) + R\bar{\theta}\bar{x} \cos \gamma_0 v(t) \right) - \bar{K}_r \bar{y}^2 v(t) \right\} ds \qquad (9b)$$

Defining the following modal integrals:

$$\bar{K}_x = \int \bar{K}_r \bar{x}^2 ds$$

$$S_{xy} = \int \bar{x}\bar{y} ds$$

$$S_{\theta y} = \int \bar{\theta}\bar{y} ds$$

$$\bar{K}_y = \int \bar{K}_r \bar{y}^2 ds$$

$$S_{\alpha x} = \int \bar{\alpha}\bar{x} ds$$

Then substituting into (9a) and 9(b) yields:

$$\bar{f}_h = D\dot{\psi}^2 R^2 \gamma_0 \left\{ S_{xy} \sin \gamma_0 v(t) - R S_{\alpha x} \cos \gamma_0 h(t) \right\} - \bar{K}_x h(t) \qquad (10a)$$

$$\bar{f}_v = D\dot{\psi}^2 R^2 \gamma_0 \left\{ -S_{xy} \sin\gamma_0 h(t) - RS_{\theta x} \cos\gamma_0 v(t) \right\} - \overline{K}_y v(t) \tag{10b}$$

The following variables are defined to ease the required algebraic manipulations:

$$S_0 = D\dot{\psi}^2 R^2 \gamma_0 S_{xy} \sin\gamma_0$$

$$S_h = D\dot{\psi}^2 R^3 \gamma_0 S_{\alpha y} \cos\gamma_0$$

$$S_v = D\dot{\psi}^2 R^3 \gamma_0 S_{\theta y} \cos\gamma_0$$

Substituting into (10a) and (10b) the modal rub force equations are:

$$\bar{f}_h = S_0 v(t) - \overline{K}_x h(t) - S_h h(t) \tag{11a}$$

$$\bar{f}_v = -S_0 h(t) - \overline{K}_y v(t) - S_v v(t) \tag{11b}$$

The Modal Equations of Motion

The modal equations of motion are obtained by combining the modal rub forces, (11a) and (11b), with the modal inertia, stiffness, damping and gyroscopic forces. The latter forces are obtained by using the translational and rotational values of the horizontal and vertical modal vectors to transform the physical inertia, stiffness, damping and gyroscopic properties to modal coordinates. Performing these transformations and using the modal rub forces derived in the previous section yields the following modal equations of motion (see also References [2] and [3]):

$$\ddot{h} + \omega_h^2 h + 2\omega_h \xi_h \dot{h} + 2\dot{\psi}g\dot{v} = \bar{f}_h \tag{12a}$$

$$\ddot{v} + \omega_v^2 v + 2\omega_v \xi_v \dot{v} - 2\dot{\psi}g\dot{h} = \bar{f}_v \tag{12b}$$

Substituting the rub forces and collecting coefficients of the same dependent variables we have the following:

$$\ddot{h} + \left(\omega_h^2 + \overline{K}_x + S_h\right)h + 2\omega_h \xi_h \dot{h} + 2\dot{\psi}g\dot{v} - S_0 v = 0 \tag{13a}$$

$$\ddot{v} + \left(\omega_v^2 + \overline{K}_y + S_v\right)v + 2\omega_v \xi_v \dot{v} - 2\dot{\psi}g\dot{h} + S_0 v = 0 \tag{13b}$$

In general, the system is non-axisymmetric in the rotor/support modal stiffness, as well as in the damping. Also, unless the horizontal and vertical modal displacements are equal, the direct rub force coefficients S_h, and S_v, $\overline{K}_x$ and $\overline{K}_y$, also are not equal. However, many incidents of turbine engine whirling exhibit circular or almost circular orbits. Furthermore, one may also be able to construct highly elliptical orbits from identical uncoupled horizontal and vertical mode shapes even when the corresponding modal stiffnesses or frequencies are grossly different. Therefore, to simplify the modal equations of motion it will be assumed that the horizontal and vertical mode shapes are the same. However, the support stiffnesses and modal frequencies may be unequal.

With this consideration the following additional simplification is made:

$$S_h = S_v = S_1$$
$$\overline{K}_x = \overline{K}_y = \overline{K}$$

Where the local case and rotor radial stiffness is assumed to be isotropic and axisymmetric. Additionally, the modal damping is assumed to be identical in both planes. The damping symmetry is justified since damping values are difficult to obtain and in most analyses is an educated guess, at best. Assuming symmetric damping:

$$\zeta = 2\omega_h \xi_h = 2\omega_v \xi_v$$

Finally, the modal equations of motion are:

$$\ddot{h} + \left(\omega_h^2 + \overline{K} + S_1\right)h + \zeta\dot{h} + 2\dot{\psi}g\dot{v} - S_0 v = 0 \qquad (14a)$$

$$\ddot{v} + \left(\omega_v^2 + \overline{K} + S_1\right)v + \zeta\dot{v} - 2\dot{\psi}g\dot{h} + S_0 h = 0 \qquad (14b)$$

Where the support non-axisymmetry is reflected in the horizontal and vertical modal frequencies, ω_h and ω_v.

Stability of Rotor with Axisymmetric Supports

It had been shown by various investigators such as Gallardo [4], Smith [5], Ehrich [6], Alford [7] and others, that the least stable rotating system is axisymmetric. Thus one may initially perform a stability analysis of a rotor with an axisymmetric support. Then, if the axisymmetric system is stable, no additional analysis may be required, since a non-axisymmetric supported system will always be more stable than the identical rotor with symmetric supports.

In the special problem of an axisymmetric rotor system the stability/instability conditions are obtained with greater simplicity and directness. The modal equations of motion of rotor with axisymmetric support stiffness are obtained from the previous section by making the modal frequencies equal. Thus:

$$\ddot{h} + \left(\omega_0^2 + \overline{K} + S_1\right)h + \zeta\dot{h} + 2\dot{\psi}g\dot{v} - S_0 v = 0 \tag{15a}$$

$$\ddot{v} + \left(\omega_0^2 + \overline{K} + S_1\right)v + \zeta\dot{v} - 2\dot{\psi}g\dot{h} + S_0 h = 0 \tag{15b}$$

Where: $\omega_0 = \omega_h = \omega_v$.

Equations (15a) and (15b) are combined into a single equation by introducing the complex variable z, where $z = h + iv$. Multiplying the second equation of motion, (15b), by (i) and adding it to the first, (15a), and using the complex variable definition, we have

$$\ddot{z} + \left(\omega_0^2 + \overline{K} + S_1\right)z + \zeta\dot{z} - i2\dot{\psi}g\dot{z} + iS_0 z = 0 \tag{16}$$

Assuming a solution in the complex exponential form:

$$z = z_0 e^{i\Lambda t} = z_0 e^{i(A+iB)t} = z_0 e^{-Bt}e^{iAt}$$

Where the coefficient z_0 and the eigenvalue Λ are complex quantities and "A" and "B" are real numbers.

Furthermore:

Real $(\Lambda) = A$ = frequency (real quantity)

Imag $(\Lambda) = B$ = attenuation or amplification factor (real quantity) and:

B > 0, stable

B < 0, unstable

Stability/Instability Boundary of the Axisymmetric System

Substituting the exponential solution into the complex variable equation of motion (16), one obtains:

$$\left\{\left\langle\Lambda^2 - 2\Lambda\dot{\psi}g - \left(\omega_0^2 + \overline{K} + S_1\right)\right\rangle - i\left\langle\Lambda\zeta + S_0\right\rangle\right\}z = 0 \tag{17}$$

The condition for the existence of a nontrivial solution requires that the polynomial coefficient of "z" must vanish:

$$F(\Lambda) = \left\{\left\langle\Lambda^2 - 2\Lambda\dot{\psi}g - \left(\omega_0^2 + \overline{K} + S_1\right)\right\rangle - i\left\langle\Lambda\zeta + S_0\right\rangle\right\} = 0 \tag{18}$$

At the stability-instability boundary, the solution is simple harmonic, which implies that Λ = Real at the boundary. Therefore, since all the parameters in the Eigen polynomial, (18), are real and non-zero, then the following holds at the stability/instability boundary:

$$\text{Real part of } F(\Lambda) = 0$$

$$\text{Imaginary part of } F(\Lambda) = 0$$

And we have two equations at the stability boundary:

$$\Lambda^2 - 2\Lambda\dot{\psi}g - \left(\omega_0^2 + \overline{K} + S_1\right) = 0 \qquad (19a)$$

$$\Lambda\zeta + S_0 = 0 \qquad (19b)$$

Equation (19a) is solved for Λ which gives the possible frequencies at the boundary.

$$\Lambda = \dot{\psi}g \pm \sqrt{\dot{\psi}^2 g^2 + \left(\omega_0^2 + \overline{K} + S_1\right)}$$

Using the frequency, Λ determined above and Equation (19b), the damping required for stability is:

$$\zeta_{REQ} = -\frac{S_0}{\Lambda}$$

Since the term S_0 is always destabilizing ($S_0 > 0$), for a system to be stable, the system damping ζ must be positive ($\zeta > 0$). Therefore: $\Lambda = \Lambda_- < 0$, at the boundary because $B = 0$ at the boundary, and ζ and S_0 are positive. Furthermore, $\Lambda = \Lambda_+ > 0$, is always stable because if $\Lambda > 0$ then $B > 0$.

From the above, the results for the rotor-rub stability/instability boundary are:

$$\text{Whirl frequency: } \Lambda_- = \dot{\psi}g - \sqrt{\dot{\psi}^2 g^2 + \left(\omega_0^2 + \overline{K} + S_1\right)}$$

$$\text{And the required damping, } \zeta_{REQ} = -\frac{S_0}{\Lambda_-}$$

Comparison of the available damping to the required damping (in the presence of a destabilization) gives a measure of the stability of the system:

$$\zeta_{\text{AVAILABLE}} > \zeta_{\text{REQ}}, \text{ stable}$$

$$\zeta_{\text{AVAILABLE}} < \zeta_{\text{REQ}}, \text{ unstable}$$

Recall that S_0 and S_1 contain the plowing rub coefficients, comprised of the rub constant, rotor speed, blade tip angle and tip radius and $\overline{K}_r$ is the radial-rub restoring or snubbing force, which may or may not be dependent on the rub parameters. Recognizing:

$$\cot \gamma_0 = \frac{\cos \gamma_0}{\sin \gamma_0}$$

S_1 may be written in terms of S_0 as:

$$S_1 = S_0 R p, \quad \text{where } p = \frac{S_{ax}}{S_{xy}} \cot \gamma_0$$

Making the above substitutions into the expressions for the whirl frequency and required damping, we have, at the stability instability boundary, the following results.

$$\text{Whirl frequency: } \lambda_- = \dot{\psi}g - \sqrt{\dot{\psi}^2 g^2 + \left(\omega_0^2 + \overline{K} + S_0 R p\right)} \qquad (20a)$$

$$\text{Required damping: } \zeta_{\text{REQ}} = -\frac{S_0}{\dot{\psi}g - \sqrt{\dot{\psi}^2 g^2 + \left(\omega_0^2 + \overline{K} + S_0 R p\right)}} \qquad (20b)$$

The rotor shown in Figure 1 is used to demonstrate the use of Equation (20) for determining the whirl frequency and required damping for a known quantity of rub. The rotor consists of a massless cantilever shaft with a disk attached at the free end. The properties of the rotor are shown in the figure. The non-rotating frequencies and modes shapes are computed from a finite element analysis. The first modal frequency and mode shape is:

$$\omega_0 = \omega_h = \omega_v = 3.071 \times 10^2 \text{ rad/sec}$$

$$\Phi = \left\langle \begin{array}{c} \overline{x} \\ \overline{a} \end{array} \right\rangle = \left\langle \begin{array}{c} \overline{y} \\ \theta \end{array} \right\rangle = \left\langle \begin{array}{c} 1.830 \\ 7.144 \times 10^{-2} \end{array} \right\rangle$$

Disk

M = 100. lbm

I_p **= 20,000. lbm-inch²**

Gravity = 386. inch/sec²

Radius = 5 inch

Blade Angle = 30 degrees

Rotation Vector

Rotor Shaft

E = 30.x10⁶ lb/inch²

I = 20. inch⁴

L = 40. inch

Rub Spring
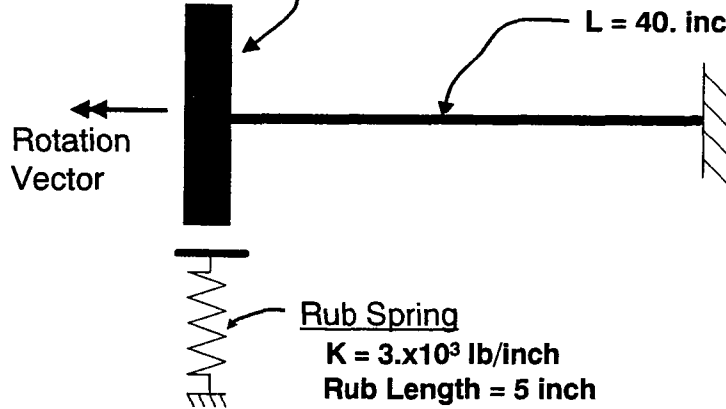
K = 3.x10³ lb/inch

Rub Length = 5 inch

Figure 1. Cantilever Rotor

Substituting the above values, and the values for blade angle, disk radius, rub stiffness and length, into Equation (20a) and (20b), the whirl frequency and required damping are:

$$\lambda = 0.2644\,\dot{\psi} - / + \sqrt{.0699\,\dot{\psi}^2 + 14.45 \times 10^4 + 37.04\,D\,\dot{\psi}^2}$$

$$\zeta_{REQ} = -\frac{109.6\,D\,\dot{\psi}^2}{\lambda}$$

Figure 2 shows the result of plotting the whirl frequency and required damping as a function of rotor speed for a variety of dynamic rub coefficients. The whirl frequencies associated with both the forward, λ_+ and the backward, λ_- mode are shown in Figure 2a, however only the backward whirl frequency is significant since the backward whirl frequency is used to calculate the required damping for a stable system. The required damping as a function of rotor speed for various levels of rub is shown in Figure 2b. As expected and depicted in the figure, the required damping increases with both rotor speed and rub force.
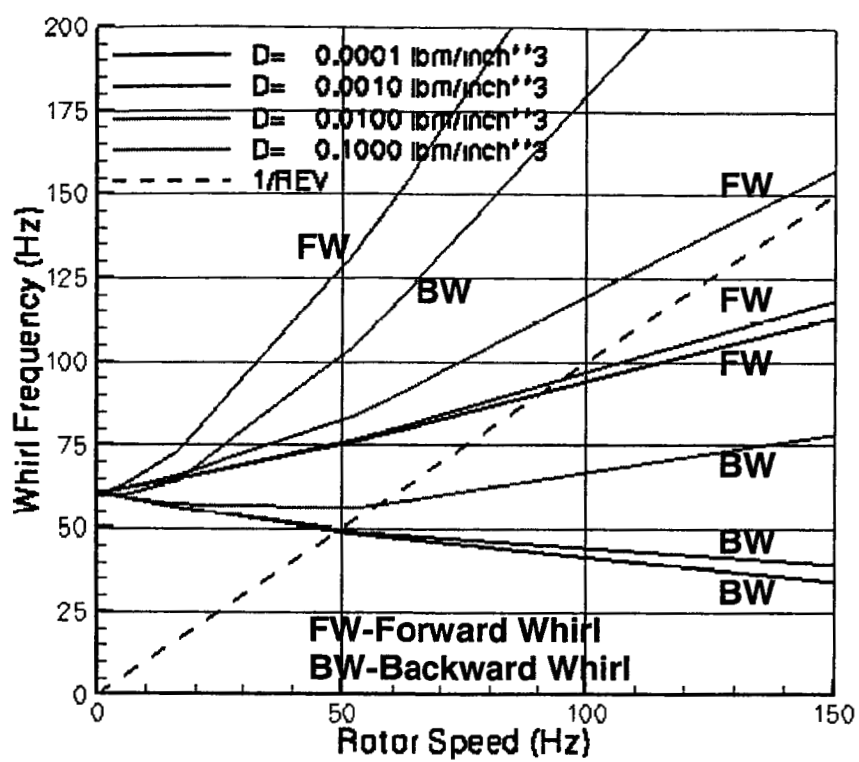
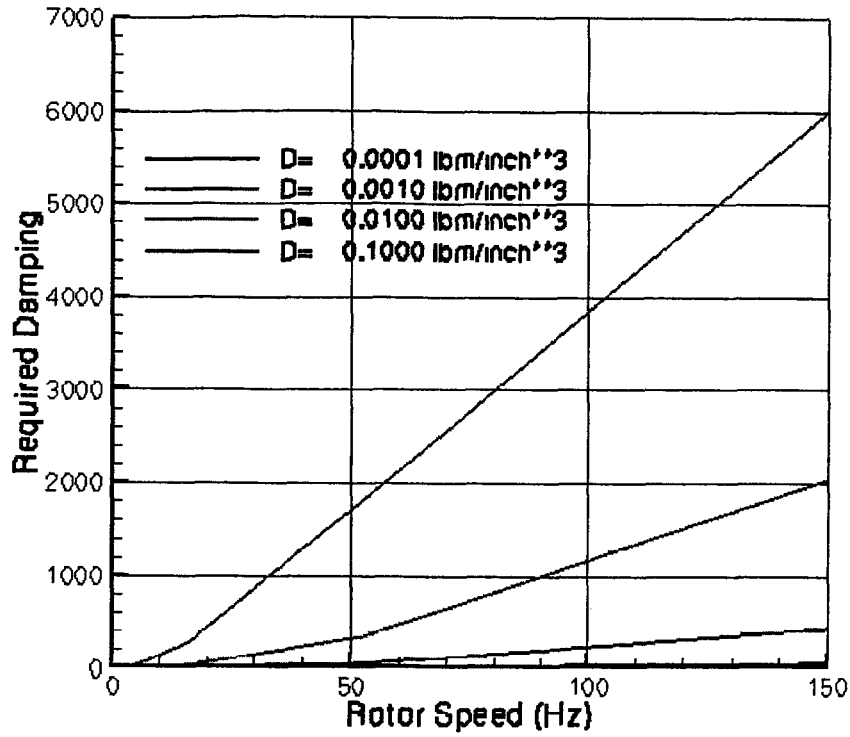**Figure 2a. Whirl Frequency at Stability/Instability Boundary**

Figure 2b. Required Damping at Stability/Instability Boundary

Allowable Destabilization for an Available Damping

The damping relation may be solved for the allowable destabilization S_0, in terms of the available damping. In this context, Equation (20b) is solved (with the available damping assumed to be known), for the destabilization that can be allowed that will make the initially stable system approach neutral stability or the stability/instability boundary.

Thus the corresponding relation is:

$$S_0 = -\lambda\zeta$$

And the converse expression for the allowable destabilization in the presence of the available damping becomes,

$$S_{0_{Allowable}} = -\zeta_{Available}\left\{\dot{\psi}g - \sqrt{\dot{\psi}^2 g^2 + \left(\omega_0^2 + \overline{K} + S_{0_{Allowable}}Rp\right)}\right\} \qquad (21)$$

Equation (21) is complicated since S_0 is on both sides of the equal (=) sign. However, with some algebraic manipulation, the allowable destabilization can be obtained explicitly. Rearranging Equation (21) yields:

$$S_{0_{Allowable}} = \frac{\zeta(\zeta Rp - 2\dot{\psi}g)}{2} \pm \zeta\sqrt{\left(\frac{\zeta Rp - 2\dot{\psi}g}{2}\right)^2 + \left(\omega_0^2 + \overline{K}\right)} \tag{22}$$

Noting that S_0 occurs skew-symmetrically in the modal equations of motion [Equation (13)], and that it has a positive sign (+) in the vertical modal equation and a negative (-) sign in the horizontal. Thus, in the mathematical context, only a positive (+) for S_0 is admissible. The signs of S_0 in the equations of motion reflect the unstable whirl direction, such that a positive sign in the vertical DOF (negative in the horizontal DOF) may indicate a backward whirl, while opposite signs would denote a forward whirl. This effect is also seen in the expression for the required damping.

The assessment of system stability or instability can then be made by comparing the available destabilization to its allowable value, in the presence of the available damping. Thus:

$$S_{0_{Available}} \langle S_{0_{Allowable}}, \text{ stable}$$

$$S_{0_{Available}} \rangle S_{0_{Allowable}}, \text{ unstable}$$

Concluding Remarks

The linearized modal equations of motion of a rotor with non-axisymmetric stiffnesses with both gyroscopic and rub forces have been obtained. The equations of the special case of a rotor with axisymmetric support have also been derived. The latter equations are used to determine the influence of the rub forces on rotor stability/instability boundaries.

The two-mode representation (one in each vertical and horizontal plane) of the equations of motion was considered. For the axisymmetric support case, the modal equations of motion were reduced to one equation using a complex variable to describe the rotor motions. A method was derived to investigate the conditions at the stability/instability boundary.

At the boundary, the whirl frequencies and required damping for the available value of the de-stabilizing coefficient are obtained. The converse, which is the allowable de-stabilizing coefficients in terms of the available damping, is also derived. These expressions are in closed form, and provide a convenient way to assess the stability/instability implications of both rub force parameters and the rotor system's modal inertial, damping, elastic and gyroscopic properties.

The analysis for the axisymmetric rotor system may be considered as a negative stability/instability criterion, in that an axisymmetric system is the least stable configuration.

Thus, if this analysis indicates a stable rotor, an analysis of a rotor with non-axisymmetric support stiffnesses may not be required, since the latter configuration is always more stable than the former.

References

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